Lecture 39

Quantum Coherent State of Light

As mentioned in the previous lecture, the discovery of Schrödinger wave equation was a resounding success. When it was discovered by Schrödinger, and appplied to a very simple hydrogen atom, its eigensolutions, especially the eigenvalues E_n coincide beautifully with spectroscopy experiment of the hydrogen atom. Since the electron wavefunctions inside a hydrogen atom does not have a classical analog, less was known about these wavefunctions. But in QED and quantum optics, the wavefunctions have to be connected with classical electromagnetic oscillations. As seen previously, electromagnetic oscillations resemble those of a pendulum. The original eigenstates of the quantum pendulum were the photon number states also called the Fock states. The connection to the classical pendulum was tenuous, but required by the correspondence principle–quantum phenomena resembles classical phenomena in the high energy limit. This connection was finally established by the establishment of the coherent state.

39.1 The Quantum Coherent State

We have seen that the photon number states[1] of a quantum pendulum do not have a classical correspondence as the average or expectation values of the position and momentum of the pendulum are always zero for all time for this state. Therefore, we have to seek a time-dependent quantum state that has the classical equivalence of a pendulum. This is the coherent state, which is the contribution of many researchers, most notably, Roy Glauber (1925–2018) [286] in 1963, and George Sudarshan (1931–2018) [287]. Glauber was awarded the Nobel prize in 2005.

We like to emphasize again that the modes of an electromagnetic cavity oscillation are homomorphic to the oscillation of a classical pendulum. Hence, we first connect the oscillation of a quantum pendulum to a classical pendulum. Then we can connect the oscillation of a

[1] In quantum theory, a "state" is synonymous with a state vector or a function.

classical electromagnetic mode to that of a quantum electromagnetic mode, and then the connection between the classical pendulum to the quantum pendulum. The coherent state is a linear superposition of photon number states that makes it look like a wave packet. A photon number state does not resemble a wave packet, and hence, it does not resemble a classical pendulum in the correspondence-principle limit. As shall be shown, a coherent state can make a quantum pendulum resemble a classical pendulum in the correspondence-principle limit.

39.1.1 Quantum Harmonic Oscillator Revisited—Creation and Annihilation Operators

To this end, we revisit the quantum harmonic oscillator or the quantum pendulum with more mathematical depth so as to develop the necessary tools for deriving the quantum coherent state. Rewriting Schrödinger equation as the eigenequation for the photon number state for the quantum harmonic oscillator, we have

$$\hat{H}\psi(x) = \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega_0^2 x^2\right]\psi(x) = E\psi(x). \tag{39.1.1}$$

where $\psi(x)$ is the eigenfunction, and E is the eigenvalue. The above can be changed into a dimensionless form first by dividing $\hbar\omega_0$, and then let $\xi = \sqrt{\frac{m\omega_0}{\hbar}}x$ be a normalized dimensionless variable. The above then becomes

$$\frac{1}{2}\left(-\frac{d^2}{d\xi^2} + \xi^2\right)\psi(\xi) = \frac{E}{\hbar\omega_0}\psi(\xi) \tag{39.1.2}$$

We can define normalized variables $\hat{\pi} = -i\frac{d}{d\xi}$ and $\hat{\xi} = \hat{I}\xi$ to rewrite the Hamiltonian as

$$\hat{H} = \frac{1}{2}\hbar\omega_0(\hat{\pi}^2 + \hat{\xi}^2) \tag{39.1.3}$$

Furthermore, the Hamiltonian in (39.1.2) looks almost like $A^2 - B^2$, and hence motivates its factorization. To this end, we first show that

$$\frac{1}{\sqrt{2}}\left(-\frac{d}{d\xi} + \xi\right)\frac{1}{\sqrt{2}}\left(\frac{d}{d\xi} + \xi\right) = \frac{1}{2}\left(-\frac{d^2}{d\xi^2} + \xi^2\right) - \frac{1}{2}\left(\frac{d}{d\xi}\xi - \xi\frac{d}{d\xi}\right) \tag{39.1.4}$$

It can be shown easily that as operators (meaning that they will act on a function to their right), the last term on the right-hand side is an identity operator, namely that

$$\left(\frac{d}{d\xi}\xi - \xi\frac{d}{d\xi}\right)f(\xi) = \hat{I}f(\xi) = f(\xi) \tag{39.1.5}$$

Therefore, the last term in (39.1.4) is an identity operator, and (39.1.4) becomes

$$\frac{1}{2}\left(-\frac{d^2}{d\xi^2} + \xi^2\right) = \frac{1}{\sqrt{2}}\left(-\frac{d}{d\xi} + \xi\right)\frac{1}{\sqrt{2}}\left(\frac{d}{d\xi} + \xi\right) + \frac{1}{2} \tag{39.1.6}$$

We now define a new operator

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{d\xi} + \xi\right) \tag{39.1.7}$$

The above is the creation, or raising operator and the reason for its name is obviated later. Moreover, we define

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{d}{d\xi} + \xi\right) \tag{39.1.8}$$

which represents the annihilation or lowering operator.[2] With the above definitions of the raising and lowering operators, it is easy to show that by straightforward substitution that

$$\left[\hat{a}, \hat{a}^\dagger\right] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = \hat{I} \tag{39.1.9}$$

Therefore, Schrödinger equation (39.1.2) for a quantum harmonic oscillator can be rewritten more concisely as

$$\frac{1}{2}\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right)\psi = \left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\psi = \frac{E}{\hbar\omega_0}\psi \tag{39.1.10}$$

In mathematics, a function is analogous to a vector. So ψ is the implicit representation of a vector. The operator

$$\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

is an implicit[3] representation of an operator, and in this case, a differential operator. So in the above, namely (39.1.10), is analogous to the matrix eigenvalue equation $\overline{\mathbf{A}} \cdot \mathbf{x} = \lambda\mathbf{x}$.

Consequently, the Hamiltonian operator can now be expressed concisely as

$$\hat{H} = \hbar\omega_0\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \tag{39.1.11}$$

Equation (39.1.10) above is in implicit math notation. In implicit Dirac notation, it is

$$\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|\psi\rangle = \frac{E}{\hbar\omega_0}|\psi\rangle \tag{39.1.12}$$

In the above, $\psi(\xi)$ is a function which is a vector in a functional space. It is denoted as ψ in math notation and $|\psi\rangle$ in Dirac notation. This is also known as the "ket". The conjugate transpose of a vector in Dirac notation is called a "bra" which is denoted as $\langle\psi|$. Hence, the inner product between two vectors is denoted as $\langle\psi_1|\psi_2\rangle$ in Dirac notation.[4]

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[2]We can prove that $\hat{a}$ and $\hat{a}^\dagger$ are conjugate transpose of each other by using the definition in Section 22.2.3.

[3]A notation like $\overline{\mathbf{A}} \cdot \mathbf{x}$, we shall call implicit, while a notation $\sum_{i,j} A_{ij}x_j$, we shall call explicit.

[4]There is a one-to-one correspondence of Dirac notation to matrix algebra notation. $\hat{A}|x\rangle \leftrightarrow \overline{\mathbf{A}}\cdot\mathbf{x}$, $\langle x| \leftrightarrow \mathbf{x}^\dagger$ $\langle x_1|x_2\rangle \leftrightarrow \mathbf{x}_1^\dagger\cdot\mathbf{x}_2$. The preponderance of languages in different communities is like the story of the Tower of Babel.

If we denote a photon number state by $\psi_n(x)$ in explicit notation, ψ_n in math notation or $|\psi_n\rangle$ in Dirac notation, then we have

$$\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right)|\psi_n\rangle = \frac{E_n}{\hbar\omega_0}|\psi_n\rangle = \left(n + \frac{1}{2}\right)|\psi_n\rangle \tag{39.1.13}$$

where we have used the fact that $E_n = (n + 1/2)\hbar\omega_0$. Therefore, by comparing terms in the above, we have

$$\hat{a}^\dagger \hat{a}|\psi_n\rangle = n|\psi_n\rangle \tag{39.1.14}$$

and the operator $\hat{a}^\dagger \hat{a}$ is also known as the number operator because of the above. It is often denoted as

$$\hat{n} = \hat{a}^\dagger \hat{a} \tag{39.1.15}$$

and $|\psi_n\rangle$ is an eigenvector of $\hat{n} = \hat{a}^\dagger \hat{a}$ operator with eigenvalue n. It can be further shown by direct substitution that

$$\hat{a}|\psi_n\rangle = \sqrt{n}|\psi_{n-1}\rangle \quad \Leftrightarrow \hat{a}|n\rangle = \sqrt{n}|n-1\rangle \tag{39.1.16}$$

$$\hat{a}^\dagger|\psi_n\rangle = \sqrt{n+1}|\psi_{n+1}\rangle \Leftrightarrow \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \tag{39.1.17}$$

hence their names as lowering and raising operator.[5]

39.2 Some Words on Quantum Randomness and Quantum Observables

We saw previously that in classical mechanics, the conjugate variables p and x are deterministic variables. But in the quantum world, they become random variables with means and variance. It was quite easy to see that x is a random variable in the quantum world. But the momentum p is elevated to become a differential operator $\hat{p}$, and it is not clear that it is a random variable anymore.

Quantum theory is a lot richer in content than classical theory. Hence, in quantum theory, conjugate variables like p and x are observables endowed with the properties of mean and variance. For them to be endowed with these properties, they are elevated to become quantum operators, which are the quantum representations of these observables. The quantum operator has to operate on a quantum state. Hence, to be meaningful, a quantum state $|\psi\rangle$ has to be defined alongside these operators representing observables. They act on the quantum state: The operator together with the quantum state endow the observable with random properties.

Henceforth, we have to extend the concept of the average of a random variable to the "average" of a quantum operator. Now that we know Dirac notation, we can write the expectation value of the operator $\hat{x}$ with respect to a quantum state ψ as

$$\bar{x} = \langle x\rangle = \langle\psi|\hat{x}|\psi\rangle \tag{39.2.1}$$

[5]The above notation for a vector could appear cryptic or too terse to the uninitiated. To parse it, one can always down-convert from an abstract notation to a more explicit notation. Namely, $|n\rangle \to |\psi_n\rangle \to \psi_n(\xi)$.

The above is the elevated way of taking the "average" of an operator $\hat{x}$ which is related to the mean of the random variable x. In the above, the two-some, $\{\hat{x}, |\psi\rangle\}$ endows the quantum observable x with random properties. Here, x becomes a random variable whose mean is given by the above formula.

As mentioned before, Dirac notation is homomorphic to matrix algebra notation. The above is similar to $\boldsymbol{\psi}^\dagger \cdot \overline{\mathbf{X}} \cdot \boldsymbol{\psi} = \bar{x}$. This quantity $\bar{x}$ is always real if $\overline{\mathbf{X}}$ is a Hermitian matrix. Hence, in (39.2.1), the expectation value $\bar{x}$ is always real if $\hat{x}$ is a Hermitian operator. In fact, it can be proved that $\hat{x}$ is Hermitian in the function space that it is defined. Furthermore, $\hat{p} = -i\hbar\partial/\partial x$ is also Hermitian, and will always have read expectation value.

Furthermore, the variance of the random variable x can be derived from the quantum operator $\hat{x}$ with respect to to a quantum state $|\psi\rangle$. It is defined as

$$\sigma_x^2 = \langle\psi|(\hat{x} - \bar{x})^2|\psi\rangle = \langle\psi|\hat{x}^2|\psi\rangle - \bar{x}^2 \tag{39.2.2}$$

where σ_x is the standard deviation of the random variable x and σ_x^2 is its variance [73, 74].

The above implies that the definition of the quantum operators and the quantum states is not unique. One can define a unitary matrix or operator $\overline{\mathbf{U}}$ such that $\overline{\mathbf{U}}^\dagger \cdot \overline{\mathbf{U}} = \bar{\mathbf{I}}$. Then the new quantum state is now given by the unitary transform $\boldsymbol{\psi}' = \overline{\mathbf{U}} \cdot \boldsymbol{\psi}$. With this, we can easily show that

$$\begin{aligned}\bar{x} &= \boldsymbol{\psi}^\dagger \cdot \overline{\mathbf{X}} \cdot \boldsymbol{\psi} = \boldsymbol{\psi}^\dagger \cdot \overline{\mathbf{U}}^\dagger \cdot \overline{\mathbf{U}} \cdot \overline{\mathbf{X}} \cdot \overline{\mathbf{U}}^\dagger \cdot \overline{\mathbf{U}} \cdot \boldsymbol{\psi} \\ &= \boldsymbol{\psi}'^\dagger \cdot \overline{\mathbf{X}}' \cdot \boldsymbol{\psi}' \end{aligned} \tag{39.2.3}$$

where $\overline{\mathbf{X}}' = \overline{\mathbf{U}} \cdot \overline{\mathbf{X}} \cdot \overline{\mathbf{U}}^\dagger$ via unitary transform. Now, $\overline{\mathbf{X}}'$ is the new quantum operator representing the observable x and $\boldsymbol{\psi}'$ is the new quantum state vector.

In the previous section, we have elevated the position variable or observable ξ to become an operator $\hat{\xi} = \xi\hat{I}$. This operator is clearly Hermitian, and hence, the expectation value of this position operator is always real. Here, $\hat{\xi}$ is diagonal in the coordinate space representation, but it need not be in other Hilbert space representations using unitary transformation shown above.

39.3 Derivation of the Coherent States

As one cannot see the characteristics of a classical pendulum emerging from the photon number states, one needs another way of bridging the quantum world with the classical world. We need to find a wave-packet description of the quantum pendulum. This is the role of the coherent state: It will show the correspondence principle, with a classical pendulum emerging from a quantum pendulum when the energy of the pendulum is large. Hence, it will be interesting to see how the coherent state is derived.

The derivation of the coherent state is more math than physics. Nevertheless, the derivation is interesting. We are going to present it according to the simplest way presented in the literature. There are deeper mathematical methods to derive this coherent state like Bogoliubov transform which is outside the scope of this course.

Now, endowed with the needed mathematical tools, we can derive the coherent state simply. To say succinctly, the coherent state is the eigenstate of the annihilation operator, namely that

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \tag{39.3.1}$$

Here, we use α as an eigenvalue as well as an index or identifier of the state $|\alpha\rangle$.[6] Since the number state $|n\rangle$ is complete, the coherent state $|\alpha\rangle$ can be expanded in terms of the number state $|n\rangle$. Or that

$$|\alpha\rangle = \sum_{n=0}^{\infty} C_n |n\rangle \tag{39.3.2}$$

When the annihilation operator is applied to the above, we have

$$\hat{a}|\alpha\rangle = \sum_{n=0}^{\infty} C_n \hat{a}|n\rangle = \sum_{n=1}^{\infty} C_n \hat{a}|n\rangle = \sum_{n=1}^{\infty} C_n \sqrt{n}|n-1\rangle = \sum_{n=0}^{\infty} C_{n+1}\sqrt{n+1}|n\rangle \tag{39.3.3}$$

The last equality follows from changing the variable of summation from n to $n+1$. Equating the above with $\alpha|\alpha\rangle$ on the right-hand side of (39.3.1), then

$$\sum_{n=0}^{\infty} C_{n+1}\sqrt{n+1}|n\rangle = \alpha \sum_{n=0}^{\infty} C_n|n\rangle \tag{39.3.4}$$

By the orthonormality of the number states $|n\rangle$ and the completeness of the set,

$$C_{n+1} = \alpha C_n/\sqrt{n+1} \tag{39.3.5}$$

Or recursively

$$C_n = C_{n-1}\alpha/\sqrt{n} = C_{n-2}\alpha^2/\sqrt{n(n-1)} = \ldots = C_0\alpha^n/\sqrt{n!} \tag{39.3.6}$$

Consequently, the coherent state $|\alpha\rangle$ is

$$|\alpha\rangle = C_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{39.3.7}$$

But due to the probabilistic interpretation of quantum mechanics, the state vector $|\alpha\rangle$ is normalized to one, or that[7]

$$\langle\alpha|\alpha\rangle = 1 \tag{39.3.8}$$

[6]This notation is cryptic and terse, but one can always down-convert it as $|\alpha\rangle \to |f_\alpha\rangle \to f_\alpha(\xi)$ to get a more explicit notation with intuitive feel.

[7]The expression can be written more explicitly as $\langle\alpha|\alpha\rangle = \langle f_\alpha|f_\alpha\rangle = \int_{-\infty}^{\infty} d\xi f_\alpha^*(\xi)f_\alpha(\xi) = 1$.

Then

$$\langle\alpha|\alpha\rangle = C_0^* C_0 \sum_{n,n'}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{n'}}{\sqrt{n'!}} \langle n'|n\rangle$$
$$= |C_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = |C_0|^2 e^{|\alpha|^2} = 1 \tag{39.3.9}$$

Therefore, $C_0 = e^{-|\alpha|^2/2}$ for normalization, or that

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{39.3.10}$$

In the above, to reduce the double summations into a single summation, we have made use of $\langle n'|n\rangle = \delta_{n'n}$, or that the photon-number states are orthonormal. Also since $\hat{a}$ is not a Hermitian operator, its eigenvalue α can be a complex number.

Since the coherent state is a linear superposition of the photon number states, an average number of photons can be associated with the coherent state. If the average number of photons embedded in a coherent is N, then it can be shown that $N = |\alpha|^2$. As shall be shown, α is related to the amplitude of the quantum oscillation: The more photons there are in a coherent state, the larger $|\alpha|$ is.

39.3.1 Time Evolution of a Quantum State

The Schrödinger equation for a quantum particle can be written concisely as

$$\hat{H}|\psi\rangle = i\hbar\partial_t|\psi\rangle \tag{39.3.11}$$

The above not only entails the form of Schrödinger equation, it is the form of the general quantum state equation. Since $\hat{H}$ is time independent, the formal solution to the above is

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle \tag{39.3.12}$$

Meaning of the Function of an Operator

At this juncture, it is prudent to digress to discuss the meaning of a function of an operator, which occurs in (39.3.12): The exponential function is a function of the operator $\hat{H}$. This is best explained by expanding the pertinent function into a Taylor series, namely,

$$f(\overline{\mathbf{A}}) = f(0)\overline{\mathbf{I}} + f'(0)\overline{\mathbf{A}} + \frac{1}{2!}f''(0)\overline{\mathbf{A}}^2 + \ldots + \frac{1}{n!}f^{(n)}(0)\overline{\mathbf{A}}^n + \ldots \tag{39.3.13}$$

Without loss of generality, we have used the matrix operator $\overline{\mathbf{A}}$ as an illustration. The above series has no meaning unless it acts on an eigenvector of the matrix operator $\overline{\mathbf{A}}$, where $\overline{\mathbf{A}}\mathbf{v} = \lambda\mathbf{v}$. Hence, by applying the above equation (39.3.13) to an eigenvector $\mathbf{v}$ of $\overline{\mathbf{A}}$, we have

$$\begin{aligned} f(\overline{\mathbf{A}})\mathbf{v} &= f(0)\mathbf{v} + f'(0)\overline{\mathbf{A}}\mathbf{v} + \frac{1}{2!}f''(0)\overline{\mathbf{A}}^2\mathbf{v} + \ldots + \frac{1}{n!}f^{(n)}(0)\overline{\mathbf{A}}^n\mathbf{v} + \ldots \\ &= f(0)\mathbf{v} + f'(0)\lambda\mathbf{v} + \frac{1}{2!}f''(0)\overline{\lambda}^2\mathbf{v} + \ldots + \frac{1}{n!}f^{(n)}(0)\lambda^n\mathbf{v} + \ldots = f(\lambda)\mathbf{v} \end{aligned} \tag{39.3.14}$$

The last equality follows by re-summing the Taylor series back into a function. Applying this to an exponential function of an operator, we have, when $\mathbf{v}$ is an eigenvector of $\overline{\mathbf{A}}$, that

$$e^{\overline{\mathbf{A}}}\mathbf{v} = e^{\lambda}\mathbf{v} \tag{39.3.15}$$

Applying this to the photon number state with $\hat{H}$ being that of the quantum pendulum and that $|n\rangle$ is the eigenvector of $\hat{H}$, then

$$e^{-i\hat{H}t/\hbar}|n\rangle = e^{-i\omega_n t}|n\rangle \tag{39.3.16}$$

where $\omega_n = \left(n + \frac{1}{2}\right)\omega_0$. In particular, $|n\rangle$ is an eigenstate of the Hamiltonian $\hat{H}$ for the quantum pendulum, or that from (39.1.11) and (39.1.14)

$$\hat{H}|n\rangle = \hbar\omega_n|n\rangle = \hbar\omega_0\left(n + \frac{1}{2}\right)|n\rangle \tag{39.3.17}$$

In other words, $|n\rangle$, a shorthand notation for $|\psi_n\rangle$ in (39.1.13), is an eigenvector of $\hat{H}$.

Time Evolution of the Coherent State

Using the above time-evolution operator, then the time dependent coherent state, after using (39.3.10), evolves in time as[8]

$$|\alpha, t\rangle = e^{-i\hat{H}t/\hbar}|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n e^{-i\hat{H}t/\hbar}}{\sqrt{n!}}|n\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n e^{-i\omega_n t}}{\sqrt{n!}}|n\rangle \tag{39.3.18}$$

[8]Note that $|\alpha, t\rangle$ is a shorthand for $f_\alpha(\xi, t)$.

By letting $\omega_n = \omega_0 \left(n + \frac{1}{2}\right)$, the above can be written as

$$|\alpha, t\rangle = e^{-i\omega_0 t/2} e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\left(\alpha e^{-i\omega_0 t}\right)^n}{\sqrt{n!}} |n\rangle \tag{39.3.19}$$

Now we see that the last factor in (39.3.19) is similar to the expression for a coherent state in (39.3.10) with $\alpha \to \alpha e^{-i\omega_0 t}$. Therefore, we can express the above more succinctly by replacing α in (39.3.10) with $\tilde{\alpha} = \alpha e^{-i\omega_0 t}$ as

$$|\alpha, t\rangle = e^{-i\omega_0 t/2} |\alpha e^{-i\omega_0 t}\rangle = e^{-i\omega_0 t/2} |\tilde{\alpha}\rangle \tag{39.3.20}$$

Consequently,

$$\hat{a}|\alpha, t\rangle = \hat{a} e^{-i\omega_0 t/2} |\alpha e^{-i\omega_0 t}\rangle = e^{-i\omega_0 t/2} \left(\alpha e^{-i\omega_0 t}\right) |\alpha e^{-i\omega_0 t}\rangle = \tilde{\alpha} |\alpha, t\rangle \tag{39.3.21}$$

Therefore, $|\alpha, t\rangle$ is the eigenfunction of the $\hat{a}$ operator with eigenvalue $\tilde{\alpha}$. But now, the eigenvalue of the annihilation operator $\hat{a}$ is a complex number which is a function of time t. It is to be noted that in the coherent state in (39.3.19), the photon number states time-evolve coherently together in a manner to result in a phase shift $e^{-i\omega_0 t}$ in the eigenvalue giving rise to a new eigenvalue $\tilde{\alpha}$!

39.4 More on the Creation and Annihilation Operator

As seen in the photon-number states, the oscillation of the pendulum does not emerge in the quantum solutions to Schrödinger equation. Hence it is prudent to see if this physical phenomenon emerge with the coherent state. In order to connect the quantum pendulum to a classical pendulum via the coherent state, we will introduce some new operators. Since

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}} \left(-\frac{d}{d\xi} + \xi\right) \tag{39.4.1}$$

$$\hat{a} = \frac{1}{\sqrt{2}} \left(\frac{d}{d\xi} + \xi\right) \tag{39.4.2}$$

We can relate $\hat{a}^\dagger$ and $\hat{a}$, which are non-hermitian, to the normalized momentum operator $\hat{\pi}$ and the normalized position operator $\hat{\xi}$ previously defined which are hermitian. Then

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}} \left(-i\hat{\pi} + \hat{\xi}\right) \tag{39.4.3}$$

$$\hat{a} = \frac{1}{\sqrt{2}} \left(i\hat{\pi} + \hat{\xi}\right) \tag{39.4.4}$$

From the above, by subtracting and adding the two equations, we arrive at

$$\hat{\xi} = \frac{1}{\sqrt{2}} \left(\hat{a}^\dagger + \hat{a}\right) = \xi \hat{I} \tag{39.4.5}$$

$$\hat{\pi} = \frac{i}{\sqrt{2}} \left(\hat{a}^\dagger - \hat{a}\right) = -i\frac{d}{d\xi} \tag{39.4.6}$$

39.4.1 The Correspondence Principle for a Pendulum

Next, we shall study the normalized position operator $\hat{\xi}$ and normalized momentum operator $\hat{\pi}$ and their expectation values (average values) under the coherent state. Notice that both $\hat{\xi}$ and $\hat{\pi}$ are Hermitian operators in the above, with real expectation values in accordance to (39.2.1). With this, the average or expectation value of the position of the pendulum in normalized coordinate, ξ, can be found by taking expectation with respect to the coherent state, or

$$\langle\alpha|\hat{\xi}|\alpha\rangle = \frac{1}{\sqrt{2}}\langle\alpha|\hat{a}^\dagger + \hat{a}|\alpha\rangle \tag{39.4.7}$$

Since by taking the complex conjugate transpose of (39.3.1)[9]

$$\langle\alpha|\hat{a}^\dagger = \langle\alpha|\alpha^* \tag{39.4.8}$$

and (39.4.7) becomes

$$\bar{\xi} = \langle\xi\rangle = \langle\alpha|\hat{\xi}|\alpha\rangle = \frac{1}{\sqrt{2}}\left(\alpha^* + \alpha\right)\langle\alpha|\alpha\rangle = \sqrt{2}\Re e\left(\alpha\right) \neq 0 \tag{39.4.9}$$

Repeating the exercise for time-dependent case, when we let $\alpha \to \tilde{\alpha}(t) = \alpha e^{-i\omega_0 t}$, then, letting $\alpha = |\alpha|e^{-i\psi}$ yields

$$\bar{\xi}(t) = \langle\xi(t)\rangle = \langle\tilde{\alpha}(t)|\hat{\xi}|\tilde{\alpha}(t)\rangle = \frac{1}{\sqrt{2}}\left[\tilde{\alpha}^*(t) + \tilde{\alpha}(t)\right]\langle\tilde{\alpha}(t)|\tilde{\alpha}(t)\rangle = \sqrt{2}\Re e\left(\tilde{\alpha}(t)\right) \neq 0 \tag{39.4.10}$$

where for the last equality, we have made used of that $\langle\tilde{\alpha}(t)|\tilde{\alpha}(t)\rangle = 1$. Then, letting $\tilde{\alpha}(t) = \alpha e^{-i\omega_0 t}$ where α is also a complex number,

$$\bar{\xi}(t) = \langle\xi(t)\rangle = \sqrt{2}|\alpha|\cos(\omega_0 t + \psi) \tag{39.4.11}$$

In the above, we use ξ to denote the random variable. So $\langle\xi(t)\rangle$ refers to the average of the random variable ξ, or $\bar{\xi}(t)$ that is a function of time.

By the same token,

$$\bar{\pi} = \langle\pi\rangle = \langle\alpha|\hat{\pi}|\alpha\rangle = \frac{i}{\sqrt{2}}\left(\alpha^* - \alpha\right)\langle\alpha|\alpha\rangle = \sqrt{2}\Im m\left(\alpha\right) \neq 0 \tag{39.4.12}$$

For the time-dependent case, we let $\alpha \to \tilde{\alpha}(t) = \alpha e^{-i\omega_0 t}$,

$$\bar{\pi}(t) = \langle\pi(t)\rangle = -\sqrt{2}|\alpha|\sin(\omega_0 t + \psi) \tag{39.4.13}$$

Hence, we see that the expectation values of the normalized coordinate and momentum just behave like a classical pendulum. There is however a marked difference: These values are mean values with standard deviations or variances that are non-zero. Thus, they have quantum fluctuation or quantum noise associated with them. Since the quantum pendulum is

[9]Dirac notation is homomorphic with matrix algebra notation. $(\overline{\mathbf{a}} \cdot \mathbf{x})^\dagger = \mathbf{x}^\dagger \cdot (\overline{\mathbf{a}})^\dagger$.

homomorphic with the oscillation of a quantum electromagnetic mode, the amplitude of a quantum electromagnetic mode will have a mean and a fluctuation as well. Now, there are quantum noise associated with a quantum observable.

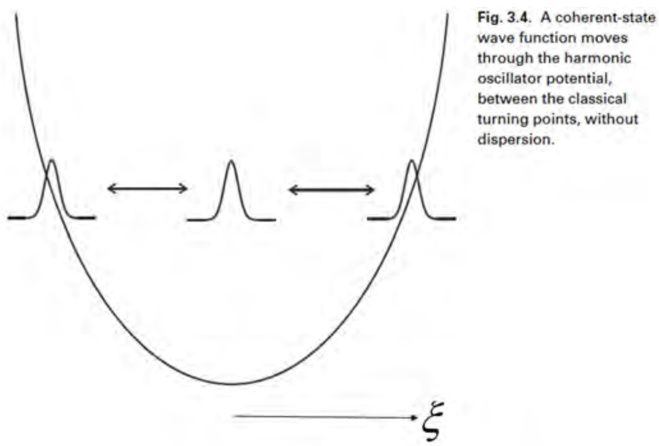

Figure 39.1: The time evolution of the coherent state. It is a wave packet that follows the motion of a classical pendulum or harmonic oscillator (courtesy of Gerry and Knight [288]).

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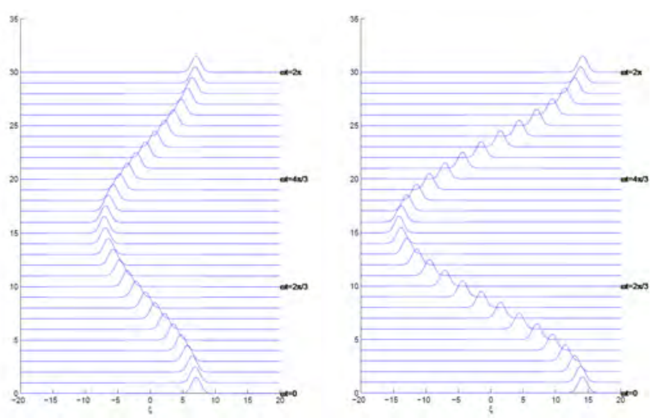

Figure 39.2: The time evolution of the coherent state for different α's. The left figure is for $\alpha = 5$ while the right figure is for $\alpha = 10$. Recall that $N = |\alpha|^2$. Again, it shows the motion of a wave packet.

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39.4.2 Connecting Quantum Pendulum to Electromagnetic Oscillator[10]

We see that the electromagnetic oscillator in a cavity is similar or homomorphic to a pendulum. The classical Hamiltonian is

$$H = T + V = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 x^2 = \frac{1}{2}\left[P^2(t) + Q^2(t)\right] = E \tag{39.4.14}$$

where E is the total energy of the system. In the above, ω_0 is the resonant frequency of the classical pendulum. To make the cavity mode homomorphic to a pendulum, we have to replace ω_0 with ω_l, the resonant frequency of the cavity mode. Or each cavity mode with resonant frequency ω_l is homomorphic with a pendulum with resonant frequency ω_0. In the above, P is a normalized momentum and Q is a normalized coordinate, and their squares have the unit of energy. We have also shown that when the classical pendulum is elevated to be a quantum pendulum, then $H \to \hat{H}$, where $\hat{H} = \hbar\omega_l\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$. Then Schrödinger equation becomes

$$\hat{H}|\psi, t\rangle = \hbar\omega_l\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|\psi, t\rangle = i\hbar\partial_t|\psi, t\rangle \tag{39.4.15}$$

Our next task is to connect the electromagnetic oscillator to this pendulum. In general, the total energy or the Hamiltonian of an electromagnetic system is

$$H = \frac{1}{2}\int_V d\mathbf{r}\left[\varepsilon\mathbf{E}^2(\mathbf{r}, t) + \frac{1}{\mu}\mathbf{B}^2(\mathbf{r}, t)\right]. \tag{39.4.16}$$

It is customary to write this Hamiltonian in terms of scalar and vector potentials. For simplicity, we use a 1D cavity, and let $\mathbf{A} = \hat{x}A_x$, $\nabla\cdot\mathbf{A} = 0$ so that $\partial_x A_x = 0$, and letting $\Phi = 0$. Then $\mathbf{B} = \nabla\times\mathbf{A}$ and $\mathbf{E} = -\dot{\mathbf{A}}$, and the classical Hamiltonian from (39.4.16) for a Maxwellian system becomes

$$H = \frac{1}{2}\int_V d\mathbf{r}\left[\varepsilon\dot{\mathbf{A}}^2(\mathbf{r}, t) + \frac{1}{\mu}\left(\nabla\times\mathbf{A}(\mathbf{r}, t)\right)^2\right]. \tag{39.4.17}$$

For the 1D case, the above implies that $B_y = \partial_z A_x$, and $E_x = -\partial_t A_x = -\dot{A}_x$. Hence, we let

$$A_x = A_0(t)\sin(k_l z) \tag{39.4.18}$$

$$E_x = -\dot{A}_0(t)\sin(k_l z) = E_0(t)\sin(k_l z) \tag{39.4.19}$$

$$B_y = k_l A_0(t)\cos(k_l z). \tag{39.4.20}$$

where $E_0(t) = -\dot{A}_0(t)$. After integrating over the volume such that $\int_V d\mathbf{r} = \mathcal{A}\int_0^L dz$, the Hamiltonian (39.4.17) then becomes

$$H = \frac{V_0\varepsilon}{4}\left(\dot{A}_0(t)\right)^2 + \frac{V_0}{4\mu}k_l^2 A_0^2(t). \tag{39.4.21}$$

[10]May be skipped on first reading.

where $V_0 = \mathcal{A}\mathcal{L}$, is the mode volume. The form of (39.4.21) now resembles the pendulum Hamiltonian. We can think of $A_0(t)$ as being related to the displacement of the pendulum. Hence, the second term resembles the "potential energy". The first term has the time derivative of $A_0(t)$, and hence, can be connected to the "kinetic energy" of the system. Therefore, we can rewrite the Hamiltonian as

$$H = \frac{1}{2}\left[P^2(t) + Q^2(t)\right] \tag{39.4.22}$$

where

$$P(t) = \sqrt{\frac{V_0\varepsilon}{2}}\dot{A}_0(t) = -\sqrt{\frac{V_0\varepsilon}{2}}E_0(t), \qquad Q(t) = \sqrt{\frac{V_0}{2\mu}}k_l A_0(t) \tag{39.4.23}$$

By elevating P and Q to be quantum operators,

$$P(t) \to \hat{P} = \sqrt{\hbar\omega_l}\hat{\pi}(t), \qquad Q(t) \to \hat{Q} = \sqrt{\hbar\omega_l}\hat{\xi}(t) \tag{39.4.24}$$

so that the quantum Hamiltonian now is

$$\hat{H} = \frac{1}{2}\left[\hat{P}^2 + \hat{Q}^2\right] = \frac{1}{2}\hbar\omega_l\left(\hat{\pi}^2 + \hat{\xi}^2\right) \tag{39.4.25}$$

similar to (39.1.3) as before, except now that the resonant frequency of this mode is ω_l instead of ω_0 because these are the cavity modes, each of which is homomorphic to a quantum pendulum of frequency ω_l. An equation of motion for the state of the quantum system can be associated with the quantum Hamiltonian just as in the quantum pendulum case.

We have shown previously that

$$\hat{a}^\dagger + \hat{a} = \sqrt{2}\hat{\xi} \tag{39.4.26}$$

$$\hat{a}^\dagger - \hat{a} = -\sqrt{2}i\hat{\pi} \tag{39.4.27}$$

Then we can let

$$\hat{P} = -\sqrt{\frac{V_0\varepsilon}{2}}\hat{E}_0 = \sqrt{\hbar\omega_l}\hat{\pi} \tag{39.4.28}$$

Finally, we arrive at

$$\hat{E}_0 = -\sqrt{\frac{2\hbar\omega_l}{\varepsilon V_0}}\hat{\pi} = \frac{1}{i}\sqrt{\frac{\hbar\omega_l}{\varepsilon V_0}}\left(\hat{a}^\dagger - \hat{a}\right) \tag{39.4.29}$$

Now that E_0 has been elevated to be a quantum operator $\hat{E}_0$, from (39.4.19), we can put in the space dependence in accordance to (39.4.19) to get

$$\hat{E}_x(z) = \hat{E}_0 \sin(k_l z) \tag{39.4.30}$$

Consequently, the fully quantized field is a field operator given by

$$\hat{E}_x(z) = \frac{1}{i}\sqrt{\frac{\hbar\omega_l}{\varepsilon V_0}}\left(\hat{a}^\dagger - \hat{a}\right)\sin(k_l z) \tag{39.4.31}$$

Notice that in the above, $\hat{E}_0$, and $\hat{E}_x(z)$ are all Hermitian operators and they correspond to quantum observables that have randomness associated with them but with real mean values. Also, the operators are independent of time because they are in the Schrödinger picture.

39.5 Epilogue

In conclusion, the quantum theory of light is a rather complex subject. It cannot be taught in just two lectures, but what we wish is to give you a peek into this theory. It takes much longer to learn this subject well: after all, it is the by product of almost a century of intellectual exercise. This knowledge is still very much in its infancy. Hopefully, the more we teach this subject, the better we can articulate, understand, and explain this subject. When James Clerk Maxwell completed the theory of electromagnetics over 150 years ago, and wrote a book on the topic, rumor has it that most people could not read beyond the first 50 pages of his book [40]. But after over a century and a half of regurgitation, we can now teach the subject to undergraduate students! When Maxwell put his final stroke to the equations named after him, he could never have foreseen that these equations are valid from nano-meter lengthscales to galactic lengthscales, from static to ultra-violet frequencies. Now, these equations are even valid from classical to the quantum world as well!

Hopefully, by introducing these frontier knowledge in electromagnetic field theory in this course, it will pique your interest enough in this subject, so that you will take this as a life-long learning experience.